2d quantum gravity with discrete edge lengths [*]

E. Bittner, H. Markum and J. Riedler [a]

[a]Institut für Kernphysik, Technische Universität Wien, Wiedner Hauptstr. 8-10, 1040 Vienna, Austria

An approximation of the Standard Regge Calculus (SRC) was proposed by the Z_2-Regge Model (Z_2RM). There the edge lengths of the simplicial complexes are restricted to only two possible values, both always compatible with the triangle inequalities. To examine the effect of discrete edge lengths, we define two models to describe the transition from the Z_2RM to the SRC. These models allow to choose the number of possible link lengths to be $n = \{4, 8, 16, 32, 64, \ldots\}$ and differ mainly in the scaling of the quadratic link lengths. The first extension, the X_n^1-Model, keeps the edge lengths limited and still behaves rather similar to the "spin-like" Z_2RM. The vanishing critical cosmological constant is reproduced by the second extension, the X_n^C-Model, which allows for increasing edge lengths. In addition the area expectation values are consistent with the scaling relation of the SRC.

The approach described in this work relies on Regge's approach to gravitation [1], a discrete description of general relativity. The theory is regularized through the introduction of a natural cutoff, the lattice spacing, to approximate space-time by a simplicial lattice. The lattice thus becomes a dynamical object, with the squared edge lengths q describing the evolution of space-time. This so-called Standard Regge Calculus (SRC) provides an interesting method to explore quantum gravity in a non-perturbative way, see [2] for a comprehensive review. Quantization of 2d SRC proceeds by evaluating the path integral

$$Z = \prod_l \int \frac{dq_l}{q_l^m} \mathcal{F}(q_l) \exp(k\sum_s \delta_s - \lambda \sum_t A_t) . \qquad (1)$$

In principle the functional integration should extend over all geometries on all possible topologies, but, as is usually done, we restrict ourselves to one specific topology, the torus. $\mathcal{F}$ is a function to assure that only Euclidean configurations of links may contribute and the real parameter m allows to choose a particular measure. The action in the exponential of (1) consists of the sum over the site-associated deficit angles δ_s times the gravitational coupling k, that is the Regge-Einstein term, and the sum over all triangle areas A_t times the cosmological constant λ. In two dimensions the

Regge-Einstein action gives the Euler characteristic of the surface and may be dropped for fixed topology.

Although the SRC code can be efficiently vectorized for large scale computing, the simulations are still a very time demanding enterprise. One therefore seeks for suitable approximations which will simplify the SRC and yet retain most of its universal features. The Z_2-Regge Model (Z_2RM) [3] could be such a desired simplification. Here the quadratic link lengths of the simplicial complexes are allowed to take on only the two values

$$q_l = 1 + \epsilon\sigma_l , \quad \sigma_l = \pm 1 . \qquad (2)$$

Then the area of a triangle t with edges q_1, q_2, q_l can be expressed as

$$\begin{aligned} A_t = \; & c_0 + c_1(\sigma_1 + \sigma_2 + \sigma_l) + c_2(\sigma_1\sigma_2 + \\ & + \sigma_1\sigma_l + \sigma_2\sigma_l) + c_3\sigma_1\sigma_2\sigma_l . \end{aligned} \qquad (3)$$

The coefficients c_i depend on ϵ only and impose the condition $\epsilon < \frac{3}{5} = \epsilon_{max}$ in order to have real and positive triangle areas, i.e. $\mathcal{F} = 1$ for all possible link length configurations. This is quite different to SRC where many potential updates either violate the triangle inequality or the manifold property. The action as well as the measure in the path integral (1) can be rewritten in terms of the Z_2-variables σ_l, see [3] for more details. Moreover, if we choose the measure parameter as

$$m = -\frac{c_1\lambda}{2\sum_{i=1}^{\infty} \frac{\epsilon^{2i-1}}{2i-1}} , \qquad (4)$$

---

the partition function of the Z_2RM takes on a particularly simple form

$$Z = \sum_{\sigma_l = \pm 1} e^{-\lambda \sum_t [c_2(\sigma_1\sigma_2 + \sigma_1\sigma_l + \sigma_2\sigma_l) + c_3\sigma_1\sigma_2\sigma_l]} .$$

To investigate the transition from the Z_2RM to SRC we allow n values for the squared edge lengths q_l

$$q_l = C(1 + \epsilon\sigma_l) , \quad 0 \le \epsilon < \epsilon_{max} , \tag{5}$$
$$\sigma_l \in \{-(n-1), -(n-1)+2, \ldots, (n-1)\} .$$

We now set the parameter $\epsilon = \frac{1}{n}$ and construct ϵ_{max} by considering the "worst case" from the point of view of the triangle inequalities. From that, it is straightforward to show

$$\epsilon_{max} = (\frac{3}{3n-1}) , \quad n \ge 2 . \tag{6}$$

What remains is to assign a role to the scaling factor C. The simplest possibility is $C = 1$, leading to an extended Z_2RM, which for example in the case $n = 8$ we call X_8^1-Regge Model. A disadvantage of the X_n^1-Regge Model (X_n^1RM) is the restriction of the link lengths to $q_l < 2$. Thus, the triangle areas are prevented to increase arbitrarily, either. In order to regain the feature of no upper boundary on the triangle areas as is manifest in SRC we set $C = \frac{n}{2}$, and call the corresponding model X_n^CRM.

In our Monte Carlo simulations we employed 16x16 lattices with periodic boundary conditions. Starting from an initial configuration consisting of equilateral triangles, 30k thermalization steps were performed. This was followed by 50k measurement steps and we actually used every 10th for analysis. Error bars were determined by the standard jackknife method using bins of 20 data.

We simulated the X_n^1RM for $n = \{2, 4, 8, 16, 32, 64, 256\}$ and analyzed expectation values of the area normalized to the total number of vertices. The SRC is known to possess an ill-defined phase for negative cosmological constants and

$$\langle A \rangle = N_1 \frac{1-m}{\lambda} \tag{7}$$

decreases inversely proportional for positive λ (N_1 is the total number of links). From the left

plots in Fig. 1 it is visible that for increasing number n of possible link lengths the critical couplings λ_c approach -4.2. The naive expectation to reproduce the transition at $\lambda = 0$ of SRC from a well to an ill-defined phase is not realized for the X_n^1RM. For $\lambda < \lambda_c$ all the triangles assume their minimum area as in the Z_2RM and do not grow unlimited like in the SRC. $\langle A \rangle$ still exhibits a decrease for positive cosmological constant λ and the functions converge for increasing n.

We simulated the X_n^CRM for $n = \{4, 8, 16, 32, 64, 256, 512\}$ and present the results in the right plots of Fig. 1. For $\lambda < 0$ the surface tries to blow up as in the SRC, all the triangle areas obtain their maximum value. The expectation values of the area approach the exact SRC result (7) for increasing number n of possible link lengths and $0 < \lambda < 1$. The deviation of $\langle A \rangle$ from the SRC result in the region $\lambda > 1$ is remarkable. The system moves into metastable states just as the SRC, but there the breathing algorithm takes care of that problem. A breathing update simply consists of rescaling all quadratic link lengths q by a factor ζ, which is the same as scaling the total area by ζ. Unfortunately a similar algorithm is not possible for the X_n^CRM. Contrary to the X_n^1RM, within the X_n^CRM the critical coupling λ_c approaches zero with increasing n. From this point of view the X_n^CRM is certainly well suited to approximate the SRC.

In order to check whether they have common universal features, too, we consider the Liouville field susceptibility

$$\chi_\phi = \langle A \rangle [\langle \phi^2 \rangle - \langle \phi \rangle^2] , \tag{8}$$

with the Liouville field $\phi = \frac{1}{A} \sum_i \ln A_i$. A is the total area and A_i the area element of the site i [4]. From continuum field theory it is known that for fixed A the susceptibility scales according to

$$\ln \chi_\phi(L) \overset{L \to \infty}{\sim} c + (2 - \eta_\phi) \ln L , \tag{9}$$

with $L = \sqrt{A}$ and the Liouville field critical exponent $\eta_\phi = 0$. This has indeed been observed for SRC with the dq/q scale invariant measure and fixed area constraint [4]. It is, however, a priori not clear whether this result will persist in the present model due to the fluctuating area and
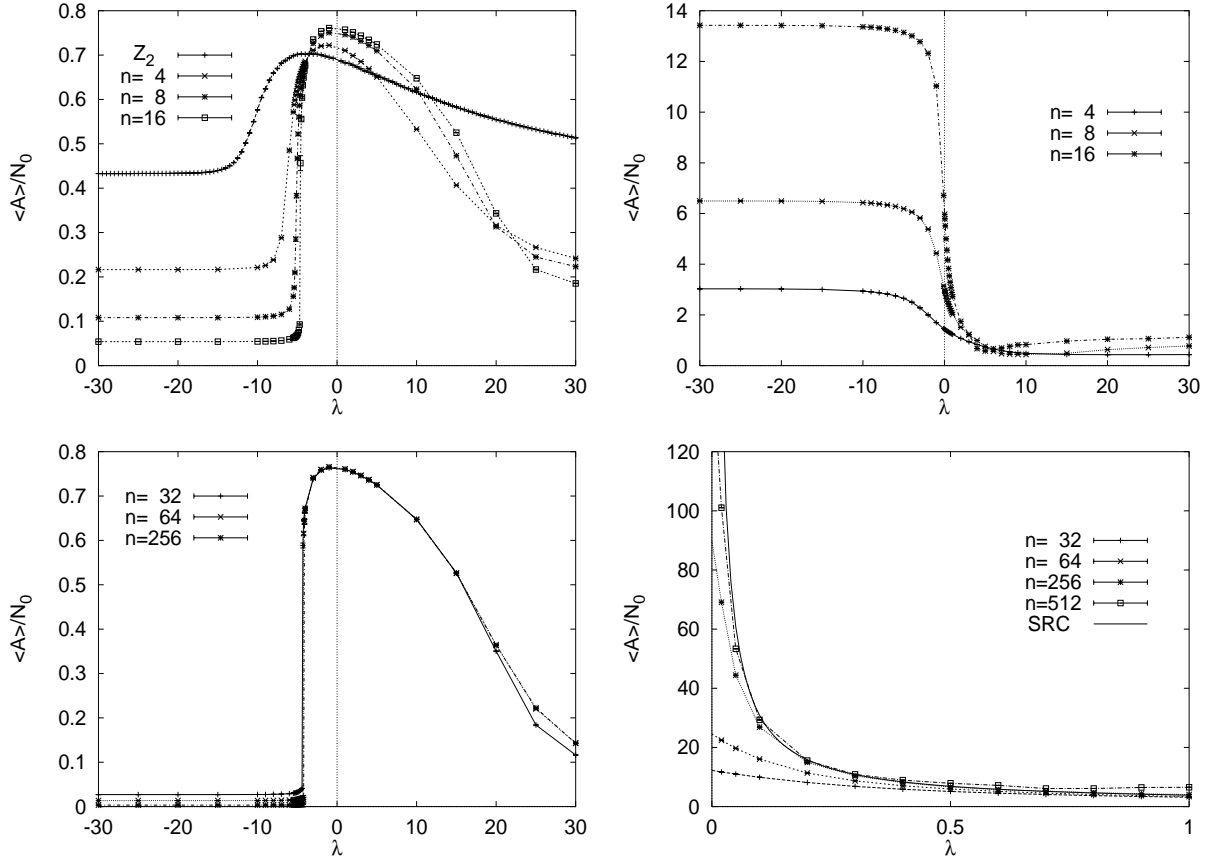
Figure 1. Expectation values of the area A normalized to the total number of vertices N_0 as a function of the cosmological constant λ for $X_n^1 \mathrm{RM}$ (left plots) and the $X_n^C \mathrm{RM}$ (right plots).

the non-scale invariant measure. Actually we find that the Liouville field susceptibility for the SRC and the $X_n^C \mathrm{RM}$ scales with $\eta_\phi \approx 2$.

In summary, to find discrete approximations of the SRC we constructed two models. The first ($X_n^1 \mathrm{RM}$) has an upper limit on the link lengths, independent of the number n they are allowed to assume, whereas in the second ($X_n^C \mathrm{RM}$) the maximum link length increases with n. Thus the $X_n^C \mathrm{RM}$ can reproduce the phase structure of SRC and shows even the same Liouville field critical exponent.

REFERENCES

1. T. Regge, Nuovo Cimento 19 (1961) 558.

2. R. Loll, Living Reviews in Relativity, to appear at http://www.livingreviews.org; gr-qc/9805049.

3. T. Fleming, M. Gross and R. Renken, Phys. Rev. D50 (1994) 7363; W. Beirl, H. Markum and J. Riedler, Int. J. Mod. Phys. C5 (1994) 359; W. Beirl, P. Homolka, B. Krishnan, H. Markum and J. Riedler, Nucl. Phys. B (Proc. Suppl.) 42 (1995) 710; E. Bittner, A. Hauke, C. Holm, W. Janke, H. Markum and J. Riedler, Nucl. Phys. B (Proc. Suppl.) 63A-C (1998) 769.

4. M. Gross and H. Hamber, Nucl. Phys. B364 (1991) 703; H. Hamber, Nucl. Phys. B (Proc. Suppl.) 99A (1991) 1; C. Holm and W. Janke, Phys. Lett. B335 (1994) 143.